

TWO-PHASE FLOW IN A DUCT OF CONSTANT CROSS SECTION (A QUALITATIVE INVESTIGATION)

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A detailed qualitative investigation of a one-dimensional stationary flow of an ideal gas carrying solid particles is carried out with the aid of a three-dimensional phase diagram. The results of the analysis allowed us to determine the character of the flow and its accompanying process of impulse and energy exchange between the phases for all regimes which are possible in principle. The boundaries of stable and unstable zones of equilibrium states are determined. It is shown that the relationships of two-phase flow in the general case (when the initial values of the velocities of the individual phases as well as temperatures are not equal) are not continuously transformed into relationships which correspond to flows with identical values of the velocity or temperature of the phases or to flows where one of the effects is absent.

The investigation of two-phase flows, as a rule, requires the use of numerical methods of calculation with the aid of a digital computer. However, the variety of possible regimes of flow and the sharp, sometimes qualitative difference of them for, apparently, closely related initial conditions make difficult the interpretation and generalization of the results of a numerical calculation [1]. Therefore a qualitative investigation, which is systematic and brought to a conclusion, of a comparatively simple (with respect to the formulation) problem concerned with a stationary one-dimensional flow in a duct of constant cross section of a mixture of gas with solid particles, with the thermal and dynamic interaction between the phases taken into account, is of interest. The difference of the local values of the velocity and temperature of the individual phases serves as a cause for the interaction. All other forms of possible effects (the friction along the wall, heat emission to the walls, and others) are not taken into account. It is assumed that the fraction of the cross section of the duct occupied by particles and their partial pressure are negligibly small. The viscosity and thermal conductivity of the gas are taken into account in implicit form only in the expressions of interaction of the phases. In addition, the Mach number calculated with respect to the relative velocity is assumed to be less than unity (i.e., the heat flux from one phase into another is approximately calculated from the difference of the values of the static temperature). It is also assumed that the diameters of the particles of the solid phase only slightly differ from one another, so that a certain effective diameter of particles can be introduced into the interaction calculation. Phase inversions, which are possible in principle, are not taken into account, but no constraints whatever are imposed on the temperature of the solid phase.

Side by side with the final results of the investigation, the method used – the analysis of the process in plane and largely in spatial (three-dimensional) phase diagrams, which is common to a number of problems of gas dynamics (for example, for MHD or EHD quasi-one-dimensional flows, flows with ionization or dissociation, and others [2-8]) – may be of interest. This applies, in particular, to the judgement expressed below about the major, qualitative (not to mention quantitative) difference of results of investigation for a flow with both forms of inequilibrium or only with one of them – dynamic or thermal. In the latter case the analysis of the problem is naturally simplified and can be confined to the framework of a plane phase diagram. However, a continuous passage to a limit for it from a three-dimensional diagram does not exist, and the conditions determining equilibrium processes and judgement about their stability are distorted. The assumption about one-dimensionality of the flow, which is common to the entire work, corresponds to the problem of qualitative investigation and, apparently, can serve as an acceptable approxi-

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TABLE 1

ξ	u_1	ρ_1	p	T_1	M^2
$k_l - 1$	2	1	γ	$(\gamma - 1)$	$-(\gamma + 1)$
$k_f - 1$	-1	1	$1 + (\gamma - 1)M^2$	$(\gamma - 1)M^2$	$-[2 + (\gamma - 1)M^2]$
$k_g - 1$	-1	1	γM	$(\gamma M^2 - 1)$	$-(\gamma M^2 + 1)$

TABLE 2

Regime	$\frac{du_1}{dx}$	$\frac{dT_1}{dx}$	$\frac{du_1}{dx}$	$\frac{dT_1}{dx}$	$\frac{du_1}{dx}$	$\frac{dT_1}{dx}$	$\frac{du_1}{dx}$	$\frac{dT_1}{dx}$
	A		B		C		D	
I $V > 1; \theta > 1$	> 0	> 0	> 0	< 0	< 0	< 0	< 0	< 0
II $V > 1; \theta < 1$	> 0	> 0	> 0	< 0	< 0	< 0	< 0	< 0
		$M < 1$		$M < 1$		$M > 1$		
III $V < 1; \theta > 1$	> 0	> 0	> 0	< 0	< 0	> 0	< 0	< 0
IV $V < 1; \theta < 1$	> 0	> 0	> 0	< 0	< 0	> 0	< 0	< 0

mation for the real phenomenon. The starting point for the investigation, just as in other analogous cases, is the equations of the law of inversion of the actions [9].

Below we present comparatively detailed reasoning and results of a physical character. As for the details of the mathematical investigation, their discussion, although somewhat sketchy owing to their cumbersome, is sufficiently complete to take into account all the basic stages of the analysis.

1. Basic Equations. For the above constraints (for a more complete form of equations of a two-phase flow see, for example, [10, 11]) the equations of continuity of motion and energy are written in the simplest form

$$\rho_1 \frac{du_1}{dx} + u_1 \frac{d\rho_1}{dx} = 0, \quad \kappa = \frac{\rho_2 u_2 F_2}{\rho_1 u_1 F_1} = \text{const} \quad (1.1)$$

$$\rho_1 u_1 \frac{du_1}{dx} + \kappa \rho_1 u_1 \frac{du_2}{dx} + \frac{dp}{dx} = 0 \quad (1.2)$$

$$\rho_1 u_1 \frac{d}{dx} \left(c_p T_1 + \frac{u_1^2}{2} \right) + \kappa \rho_1 u_1 \frac{d}{dx} \left(c T_2 + \frac{u_2^2}{2} \right) = 0 \quad (1.3)$$

where u, ρ, p, T are, respectively, the velocity, density, pressure and temperature; κ is the ratio of mass flows of the phases (solid and gaseous), F is the area of cross section of the duct (it is assumed that the area in the cross section occupied by the solid phase $F_2 \ll F_1$ and $F_1 \approx F = \text{const}$ for the gas); indices 1 and 2 here and in the following refer to the gas and particles respectively, c_p is the specific heat of the gas at a constant pressure; c is the specific heat of the particles, x is a coordinate oriented along the duct. Equations (1.1)-(1.3) must be supplemented by an equation of state of the gas; in view of the assumptions made above, this equation must be taken in the form

$$p = \rho_1 R T_1 \quad (1.4)$$

where R is the specific gas constant. We must also supplement these equations by equations which give the velocity u_2 and the temperature T_2 of the particles. With the usual assumptions, introducing the coefficients of resistance C_x and heat output α (for a particle assumed to be spherical with diameter d), the equations for the velocity and temperature of the particles are represented in the form [10-13]

$$\begin{aligned} \rho_2 u_2 \frac{du_2}{dx} &= \frac{3}{4} \frac{C_x}{d} \rho_1 |u_1 - u_2| (u_1 - u_2) \\ \rho_2 u_2 \frac{dT_2}{dx} &= \frac{6\lambda_1}{cd^2} \text{Nu} (T_1 - T_2) \end{aligned} \quad (1.5)$$

where

$$\begin{aligned} C_x &= C_x(\text{Re}), \quad \text{Re} = \frac{|u_1 - u_2| d}{\nu_1} \\ \text{Nu} &= \frac{\alpha d}{\lambda_1} = \text{Nu}(\text{Re}, \text{Pr}), \quad \text{Pr} = \frac{\rho_1 \nu_1 c_p}{\lambda_1} \end{aligned}$$

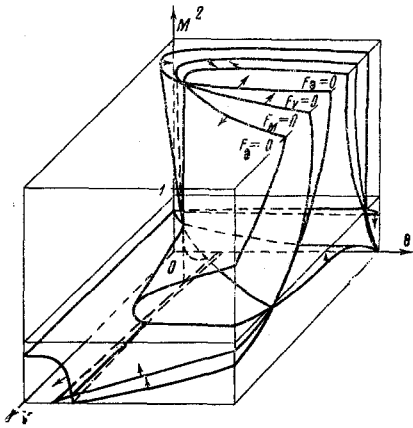


Fig. 1

The concrete criterial relationships $C_x = C_x(\text{Re})$ and $\text{Nu} = \text{Nu}(\text{Re}, \text{Pr})$ can be chosen, dependent on the range of values of the Reynolds number (see, for example, [12]). For the subsequent qualitative investigation their form is not essential. Furthermore, for what is to follow, Eqs. (1.5) are conveniently written in the form of "relaxation equations"

$$\frac{du_2}{dx} = \frac{u_1 - u_2}{l_d}, \quad \frac{dT_2}{dx} = \frac{T_1 - T_2}{l_t} \quad (1.6)$$

where l_d and l_t are respectively characteristic assumed lengths (in the general case, variable) of the dynamic and thermal relaxation.

The initial equations are transformed into the standard form of equations of inversion of actions [9] for the velocity of flow, thermodynamic parameters, and the Mach number M :

$$\begin{aligned} (M^2 - 1) &= \frac{d \ln \xi}{dx} = k_l W_l + k_f W_f + k_q W_q \\ W_l &= \kappa \frac{u_2}{a_1^2} \frac{du_2}{dx}, \quad W_f = \kappa (u_1 - u_2) \frac{\gamma}{a_1^2} \frac{du_2}{dx} \\ W_q &= -\kappa \frac{c\gamma}{c_p(\gamma - 1)} \frac{dT_2}{dx}, \quad M = \frac{u_1}{a_1}, \quad a_1^2 = \gamma \frac{p}{\rho_1} \end{aligned} \quad (1.7)$$

where ξ can be u_1 , ρ_1 , p , T_1 , or M^2 , W_l is action connected with the mechanical isentropic work of acceleration (deceleration) of the particles; W_f , W_q are actions due to friction and heat exchange between the phases, respectively; k_l , k_f , k_q are the influence coefficients presented in Table 1.

2. Preliminary Discussion. In its full form the problem of investigation reduces to the construction of the integral curves of the system of equations (1.1)-(1.4) and (1.6); i.e., it reduces to the determination of the character of variation, along the length of the duct, of the velocity and temperature of both phases u_1 , u_2 , T_1 , and T_2 as well as the pressure and the Mach number. To reduce the number of variables we introduce the relative values of the phase velocity $V = u_1/u_2$ and the temperature $\Theta = T_1/T_2$. From physical considerations it is obvious that the inequalities

$$\frac{du_2}{dx} > 0 \text{ for } V > 1, \quad \frac{du_2}{dx} < 0 \text{ for } V < 1$$

and, respectively,

$$\frac{dT_2}{dx} > 0 \text{ for } \Theta > 1, \quad \frac{dT_2}{dx} < 0 \text{ for } \Theta < 1$$

correspond to the problem under consideration.

With these constraints (including physically impossible processes of the type $du_2 > 0$ for $V < 1$ and others similar) taken into account, we can point out 16 processes which are possible in principle and which are presented in Table 2. A part of the regimes must be excluded or restricted. This applies to the regimes A-I and D-IV which contradict the first law of thermodynamics, the regime D-II (also D-IV) which contradicts the second law. In addition, proceeding from the law of inversion of actions, two of the regimes (A-II and B-II) can exist only in a subsonic region of flow, while one (C-II) can exist only in a supersonic region. (The same law excludes also the regime D-II both for $M < 1$ and for $M > 1$.) The restrictions are connected with the fact that in all regimes of D-II type the constituent elementary actions — the work done by the gas, friction and heat supply — lead to a single-valued acceleration of the gas for $M < 1$, while for $M > 1$ they lead to its deceleration and decrease in the temperature [9]. In all remaining cases (and calculation confirms this), mutual compensation of actions having an effect that is opposite in character on the gas flow is possible. These preliminary considerations limit the number of possible regimes for various values of V , Θ , and M . At the same time, the restricted regimes (A-I, D-II and D-IV) naturally are not realized for a flow in ducts of variable cross section, whereas the constraints with respect to the Mach number apply to cylindrical and narrowing ducts.

The system of equations (1.1)-(1.6) which describes the flow of a two-phase mixture contains many unknowns (u_1 , u_2 , T_1 , T_2 , p , M) whose character of variation along the length of the duct under different conditions is yet to be determined. In fact, the dependence of flow parameter on a coordinate constitutes a final result of a qualitative investigation carried out with the aid of a multidimensional phase diagram. The

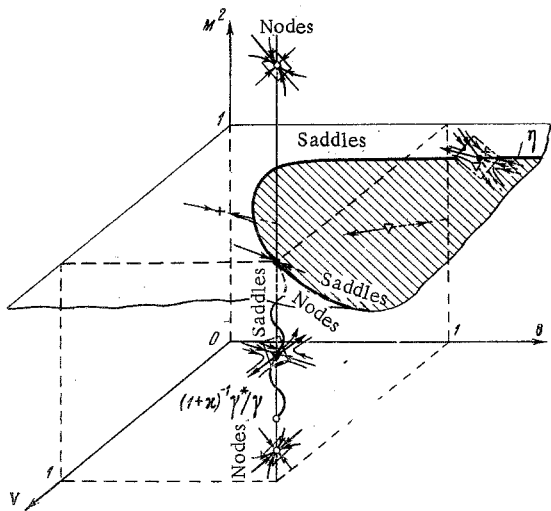


Fig. 2

they form surfaces. Such lines and surfaces are called zero lines and surfaces [2]. Zero lines (surfaces) together with straight line (plane) $M^2 = 1$,* upon passing through which the derivatives undergo a discontinuity ($\pm \infty$), divide the phase diagram into regions with constant signs of the derivatives. This allows us to represent the behavior of the trajectories at all points of the phase diagrams, with the exception of the singular points at which all zero lines (surfaces) intersect.

2) We determine the type of the singular points. The simplest singular points, as we know, are singularities of the saddle, node, focus and center types [2, 14]. In the general case, however, we encounter more complex singular points [6, 15].

3) In the phase plane we find separatrices separating one type of trajectories from another. In a phase space separatrices form separatrix surfaces.

4) We construct lines or surfaces at the points of which the derivatives of the unknowns, not being phase coordinates, become zero. From the intersection of these lines or surfaces with the trajectories we can establish the character of variation of all parameters of the flow along the duct and hence complete the investigation.

3. Three-Dimensional Phase Diagram. The system of equations of inversion of actions (1.7) for the velocity and temperature of the phases, after introduction of the relative values $V = u_2/u_1$ and $\Theta = T_1/T_2$ in the role of new dependent variables, can be reduced to the three equations

$$\begin{aligned} (M^2 - 1) \frac{dV}{V d\chi} &= \frac{F_V}{V^2 \Theta}, & (M^2 - 1) \frac{d\Theta}{\Theta d\chi} &= \frac{F_\Theta}{V^2 \Theta} \\ (M^2 - 1) \frac{dM^2}{M^2 d\chi} &= \frac{F_M}{V^2 \Theta} \end{aligned} \quad (3.1)$$

where χ is a characteristic length coordinate directed along the duct:

$$\chi = \int_0^x \frac{c}{c_p} \kappa \frac{dx}{l_i}$$

$$\begin{aligned} F_V &= (\Theta - 1) V^2 - \Theta (V - 1) \left[\gamma M^2 (V - 1) + M^2 + \frac{V^2}{\kappa} (V^2 - 1) \right] \alpha \\ F_\Theta &= -(\Theta - 1) V^2 \left[\gamma M^2 - 1 + \frac{c_p \Theta}{c \kappa} (M^2 - 1) \right] + \Theta (\gamma - 1) M^2 (V - 1) [\gamma M^2 (V - 1) + 1] \alpha \\ F_M &= (\Theta - 1) V^2 (\gamma M^2 + 1) - M^2 \Theta (V - 1) [(\gamma - 1) (\gamma M^2 + 1) (V - 1) + (\gamma + 1) V] \alpha \end{aligned}$$

and $\alpha = (c_p/c)(l_t/l_d)$. The parameter α , for the sake of simplicity, can be taken as constant. This does not influence the results of a qualitative analysis, since the type of singular points on the line of equilibrium

* The straight line (plane) $M^2 = 1$ constitutes a set of points corresponding to a limiting state of the flow. Some of them are established at the exit cross section of the duct (crisis of flow [9]), others are established at the entry to it.

solution is considerably simplified if it is possible (as it is in the case under consideration) to reduce the number of phase coordinates by appropriately choosing the defining functions. In the general case of a two-phase flow, for unequal values of the velocities of the two phases as well as the temperature, the initial system of equations (1.1)-(1.6) can be reduced to three autonomous equations and, accordingly, to a three-dimensional phase space $M^2-V-\Theta$. In particular cases when $V=1$ or $\Theta=1$ (and also for $u_2 = \text{const}$ or $T_2 = \text{const}$) we have to deal with plane phase diagrams $M^2-\Theta$ or M^2-V .

The qualitative investigation of the equations by means of phase diagrams reduces to the following:

1) We find a set of points at which the trajectories of the phase space change their direction. At these points the derivatives of phase coordinates with respect to the independent variable become zero. In a phase plane sets of such points form lines, while in a space

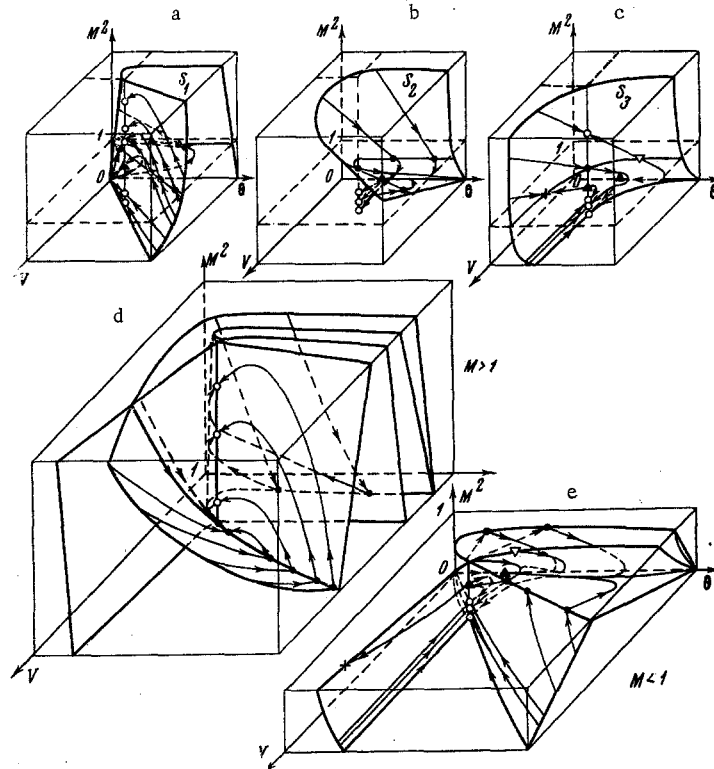


Fig. 3

does not depend on the magnitude of α (see below). With the condition $\alpha = \text{const}$ taken into account, the system of equations (3.1) is autonomous, and its solutions can be depicted by trajectories in the phase space $M^2-V-\Theta$ (of course, in the region of it where M^2 , V , and $\Theta > 0$).

The functions F_V , F_Θ and F_M being zero constitute equations of the zero surfaces, at the points of which, respectively,

$$\frac{dV}{d\chi} = 0, \quad \frac{d\Theta}{d\chi} = 0, \quad \frac{dM^2}{d\chi} = 0$$

The zero surfaces and their intersections with the plane $M^2=1$ are shown in Fig. 1. The arrows on the surfaces point to the regions of the phase space in which the sign of the derivative of the given variable with respect to the coordinate is positive. The zero surfaces intersect along the straight line $V=1$, $\Theta=1$ and along the curve $M^2 = 1$, $\eta = (\Theta - 1)V^2 - \alpha\gamma\Theta(V - 1) \cdot (V - \Gamma)$, where $\Gamma = (\gamma - 1)/\gamma$. All points of the straight line $V=1$, $\Theta=1$ correspond to the positions of equilibrium. We linearize the system (3.1) in the neighborhood of this straight line:

$$\begin{aligned} (M^2 - 1) \frac{dV}{d\chi} &= (V - 1) \alpha \frac{(1 + \kappa)}{\kappa} \left(\frac{1}{1 + \kappa} - M^2 \right) + (\Theta - 1) \\ (M^2 - 1) \frac{d\Theta}{d\chi} &= (V - 1) \alpha \gamma \Gamma M^2 - (\Theta - 1) \left[\frac{c_p}{c_x} (M^2 - 1) + \gamma M^2 - 1 \right] \\ (M^2 - 1) \frac{dM^2}{d\chi} &= - (V - 1) \alpha (\gamma + 1) M^4 + (\Theta - 1) M^2 (\gamma M^2 + 1) \end{aligned} \quad (3.2)$$

The eigenvalues which determine the type of singularity of the positions of equilibrium on the straight line $V=1$, $\Theta=1$ are found from the following determinant being zero [2, 12, 13]:

$$\begin{vmatrix} \left[\frac{\alpha(1+\kappa)}{\kappa(M^2-1)} \left(\frac{1}{1+\kappa} - M^2 \right) - \lambda \right], & \frac{1}{(M^2-1)}, & 0 \\ \alpha \frac{\gamma \Gamma M^2}{(M^2-1)}, & \left[-\frac{(M^2-1)c_p/c_x + (\gamma M^2-1)}{(M^2-1)} - \lambda \right], & 0 \\ -\alpha \frac{(\gamma+1)M^4}{(M^2-1)}, & \frac{M^2(\gamma M^2+1)}{(M^2-1)}, & -\lambda \end{vmatrix} = 0 \quad (3.3)$$

One of the eigenvalues is identically zero. Consequently, the trajectories in the neighborhood of the singular points of the straight line of equilibrium are located in planes. Two other eigenvalues $\lambda_{1,2} < 0$ for

$M^2 > 1$, $M^2 < (1 + \kappa)^{-1}\gamma^* / \gamma$ ($\gamma^* = (c_p + \kappa c) / (C_V + \kappa c)$). This corresponds to a stable position of equilibrium of the node type. The phase trajectories converge to points of this type (see Fig. 2). For small deviations from the values $V=1$ and $\Theta=1$ a tendency to return into the initial state prevails. For $(1 + \kappa)^{-1}\gamma^* / \gamma < M^2 < 1$ the eigenvalues $\lambda_1 > 0$, $\lambda_2 < 0$. On this segment of the straight line $V=1$, $\Theta=1$ singular points of the saddle type correspond to unstable positions of equilibrium. Small deviations from them lead to further widening of the trajectories away from the positions of equilibrium.

To determine the singular points in the plane $M^2=1$ we use the variable η introduced above. The system of equations (3.1) can be reduced to two differential equations in M^2 and η . Consequently, analogously to [4] the phase trajectories in the neighborhood of the singular points of the plane $M^2=1$ are also located in planes. The equation in M^2 and η after linearization has the form

$$\frac{dM^2}{d\eta} = \frac{A\eta + B(M^2 - 1)}{C\eta + D(M^2 - 1)} \quad (3.4)$$

where

$$\begin{aligned} A &= \frac{(\gamma + 1)}{\Theta V^2}, & B &= \alpha \gamma^2 \frac{(V - 1)}{V^2} \left(\frac{\gamma + 1}{\gamma} \Gamma - V \right) \\ C &= \frac{\alpha \gamma}{V} (\Gamma + 1) + 2(V - \alpha \gamma) - \frac{(\gamma + 1)}{\Theta} \\ D &= \alpha \gamma \Theta (V - 1) \left\{ \frac{\gamma V}{\Theta} \left[\gamma \Gamma (V - \Gamma) - 1 \right] - \frac{c_p \Theta}{c \kappa} (V - \Gamma) + \left(\frac{V}{\gamma \kappa} - \frac{V - \Gamma}{V} \right) \times \right. \\ &\quad \left. \times \left[\alpha \gamma (2V - \Gamma - 1) - 2V \frac{(\Theta - 1)}{\Theta} \right] \right\} \end{aligned}$$

The eigenvalues are found from the following determinant [13] being zero:

$$\begin{vmatrix} (A - \lambda) & B \\ C & (D - \lambda) \end{vmatrix} = 0$$

The eigenvalues depend on the coordinates of the singular points and the parameters of the problem. One of the possible distributions of singular points on the curve $\eta=0$, $M^2=1$ is shown in Fig. 2. For $V < 1$ a part of the curve can consist of singular points of the focus type (for example, for $\alpha=1$, $\gamma=1.4$ the region of foci is formed for the values $\kappa=0.17$). The insignificant dimension of this region and the narrow range of parameters which corresponds to it as well as the relative simplicity of the trajectories allow us to omit the analysis of flows for singular points of the focus type. A continuous transition of the velocity of flow of the gas through the velocity of sound is possible through singular points of the saddle and node types. The curve $\eta=0$ divides the plane $M^2=1$ into regions of the flow crisis (not shaded in Fig. 2) and the limiting states at the entry to the duct (shaded in Fig. 2).

The origin of the coordinates of the phase space ($M^2=V=\Theta=0$) also constitutes a singular point — a three-dimensional node. Indeed, linearizing Eqs (3.1) in the neighborhood of this point, we obtain

$$\frac{dM^2}{dV} = \frac{M^2}{V}, \quad \frac{dM^2}{d\Theta} = \frac{M^2}{\Theta}$$

Consequently, $\lambda_1 = \lambda_2 = \lambda_3$.

To construct the field of trajectories in the entire phase space we find the separatrix surfaces formed from trajectories issuing in different directions from the singular points of the straight line of equilibrium $V=\Theta=1$ and the curve $\eta=0$, $M^2=1$. Three separatrix surfaces (S_1 , S_2 and S_3) are depicted in Fig. 3. The surface S_1 (Fig. 3a) passes through the straight line of equilibrium $V_2=1$, $\Theta=1$, the Θ ($V=M^2=0$) axis, and the curve $\eta=0$, $M^2=1$. A continuous transition from subsonic to supersonic flows is basically realized along this surface. The surface S_2 passes through the curve $\eta=0$, $M^2=1$, a part of the Θ ($V=M^2=0$) axis, and a part of the straight line $\Theta=1$, $M^2=0$. A continuous transition from supersonic to subsonic flows is realized along it. Finally, the surface S_3 (Fig. 3c) passes through the straight line of equilibrium $V=1$, $\Theta=1$, the straight line $\Theta=1$, $M^2=0$ and a part of the Θ ($V=M^2=0$) axis. A trajectory passing through the point $M^2=1$, $V=1$, $\Theta=1$ belongs to this surface. The separatrix surfaces divide the phase space into six regions, each of which includes a subregion of supersonic (Fig. 3d) and that of subsonic flow of the gas (Fig. 3e). On the separatrix surfaces (Fig. 3) we have marked individual trajectories which form these surfaces. The arrows on the trajectories indicate the variation of the phase variables M^2 , Θ and V along the duct (in the positive direction of the χ coordinate).

The trajectories of the phase space which do not belong to the phase surfaces are drawn to them in the neighborhood of singular lines. This circumstance allows us to represent quite clearly the entire field

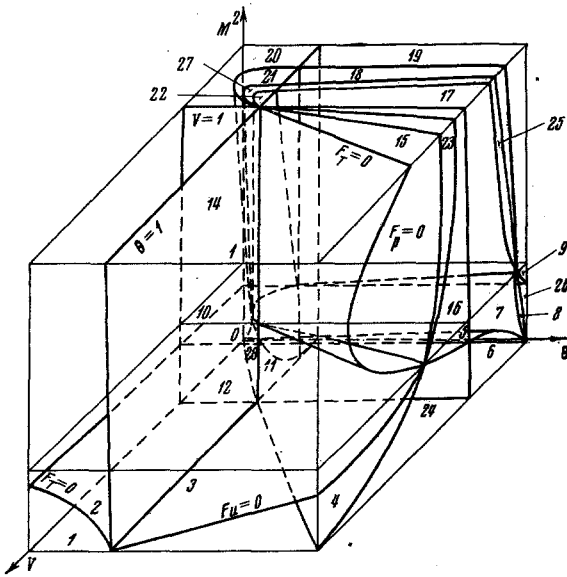


Fig. 4

for $1 > M^2 > (1 + \kappa)^{-1} \gamma^* / \gamma$ it is a maximum for the trajectories tending to the lines of equilibrium and a minimum for trajectories going away from it. In the plane $M^2=1$ the derivatives of the gas-dynamic parameters with respect to the coordinate tend to infinity, while the variation in the entropy is bounded. Therefore the states for $M^2=1$ are isentropic (in relation to the variation of the flow parameters $dS/du_1=0$ and others). The set of points bounded by the curve $\eta=0$ in the plane $M^2=1$ (shaded in Fig. 2) corresponds to states with minimum entropy, while the rest of the points of the plane correspond to a maximum of entropy. Thus, all states on the line of equilibrium $V=1, \theta=1$ and the plane $M^2=1$ (excluding the access points on the line $\eta=0, M^2=1$) are limiting states — the initial states in which the entropy is a minimum and the final states in which it is a maximum. At the access points all flow parameters, including entropy, vary continuously.

To determine the character of variation of the different variables along the length of the duct, we additionally find surfaces in the phase space at points of which the derivatives of pressure, velocity and temperature of the phases with respect to the independent parameter χ become zero. From Eqs. (1.6) and the law of inversion of actions (1.7) we obtain

$$\begin{aligned} \frac{dp}{d\chi} = 0, & \quad \text{if} \quad F_p = [V + \gamma\Gamma M^2(V-1)]\alpha(V-1)\theta - V^2(\theta-1) = 0 \\ \frac{du_1}{d\chi} = 0, & \quad \text{if} \quad F_u = (\theta-1)V^2 - \theta\gamma M^2(V-1)(V-\Gamma) = 0 \\ \frac{dT_1}{d\chi} = 0, & \quad \text{if} \quad F_T = (\theta-1)V^2(1-\gamma M^2) + \theta\gamma\Gamma M^2(V-1)[\gamma M^2(V-1)+1] = 0 \\ \frac{du_2}{d\chi} = 0, & \quad \text{if} \quad V = 1; \quad \frac{dT_2}{d\chi} = 0, \quad \text{if} \quad \theta = 1 \end{aligned}$$

The surfaces $F_p=0, F_u=0, F_T=0$, the planes $V=1, \theta=1$ and $M^2=1$, shown in Fig. 4, divide the phase space into 28 regions with given signs of the derivatives (see Table 3). Examining how the trajectories in each of the six zones of the phase space pass through these surfaces and between them, we can establish the character of variation of the pressure, velocity and temperature of the phases as well as other parameters along the trajectories.

In Fig. 5 we have presented a summary of practically all possible regimes of flow classified with respect to the types of trajectories which lead to supersonic and subsonic stable positions of equilibrium ($V=1, \theta=1$) or end with the crisis ($M^2=1$) as well as for trajectories which correspond to a continuous transition through the velocity of sound. The character of variation of the parameters along the length of the duct for regimes corresponding to singular points of the node type is shown in Fig. 5a ($M > 1$) and Fig. 5b ($M < 1$). The numbers above the curves denote the corresponding regions of the phase space (see Fig. 4). The regimes of flow shown in Fig. 5c correspond to singular points of the saddle type on the

of trajectories which begin at the origin of the coordinates and at the limiting states at the entry in the duct (on the plane $M^2=1$) or originate from positions of unstable equilibrium, i.e., coincide with the corresponding separatrices of saddles. The trajectories end, arriving at the crisis of the flow or at a position of stable equilibrium.

Any concrete flow in the duct is matched on the phase diagram by a definite segment of one of the phase trajectories whose initial point corresponds to the state of the flow in the entry section of the duct and whose final point corresponds to the state of flow in the exit section.

We note that the behavior of the integral curve can easily be explained by the second law of thermodynamics. The entropy S of a two-phase flow grows along the phase trajectories in the directions indicated by arrows. On the line of equilibrium $V=1, \theta=1$ an extremum of the entropy corresponds to each point: for $M^2 > 1$ and $M^2 < (1 + \kappa)^{-1} \gamma^* / \gamma$ it is a maximum;

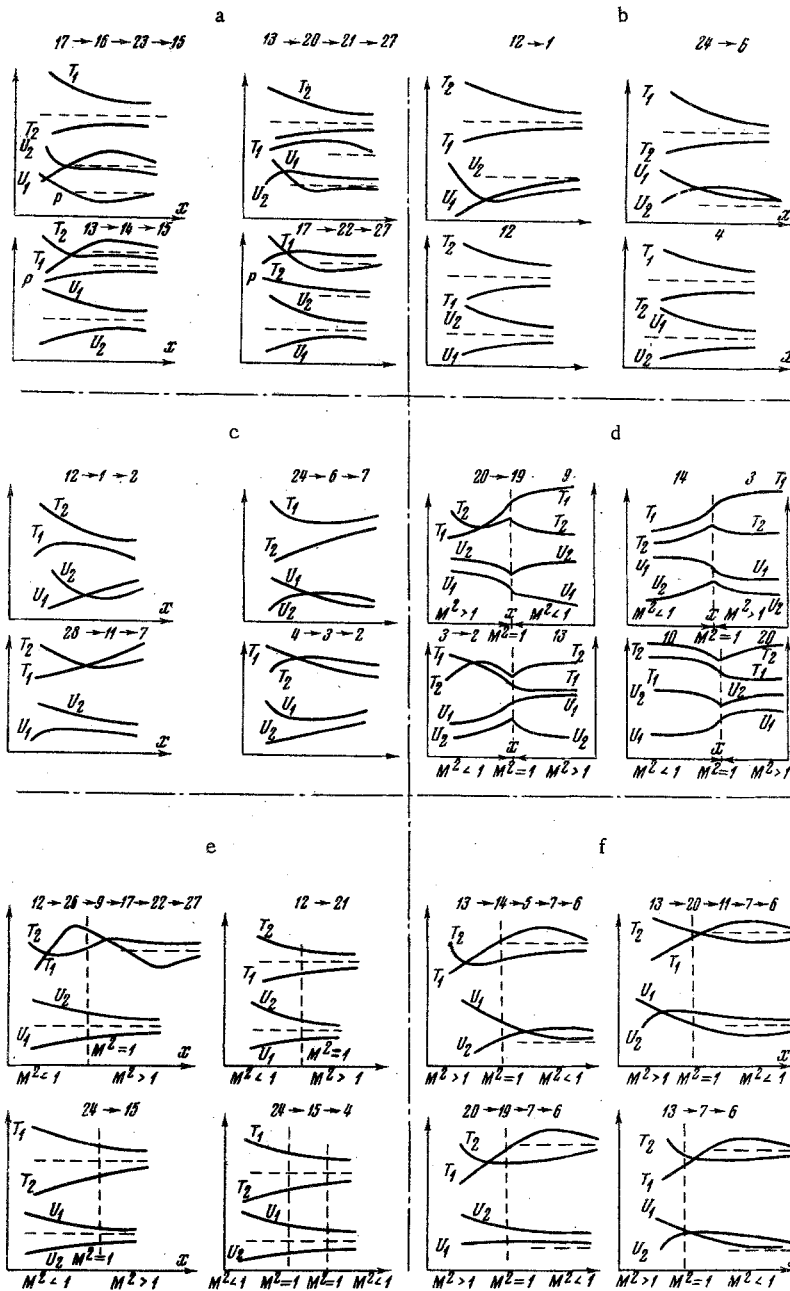


Fig. 5

straight line of equilibrium. The variation of the flow parameters ending with the crisis ($M=1$) is shown in Fig. 5d. The flows corresponding to a continuous transition through the velocity of sound along the trajectories forming the separatrix surfaces S_1 and S_2 (Fig. 3a and b) are characterized by the regimes presented in Fig. 5e and f, respectively.

4. Plane Phase Diagrams. We consider four particular cases of two-phase flows which can be investigated by means of plane phase diagrams M^2-V or $M^2-\Theta$, namely:

- 1) a two-phase flow with identical temperatures of both phases $\Theta=1$ for $V \neq 1$. Instantaneous temperature relaxation $l_t=0$ corresponds to this. Instead of Eq. (1.6) we take $dT_1/dx = dT_2/dx$;
- 2) a two-phase flow with identical phase velocities $V=1$ for $\Theta \neq 1$. The instantaneous velocity relaxation $l_d=0$ and the equation $du_1/dx = du_2/dx$ corresponds to this;
- 3) a two-phase flow with a constant temperature of the solid phase $T_2 = \text{const}$ ($dT_2/dx = 0$) in the case of infinite length of temperature relaxation $l_t = \infty$;
- 4) a two-phase flow with a constant velocity of the solid phase $u_2 = \text{const}$ ($du_2/dx = 0$) in the case $l_d = \infty$.

TABLE 3

No. of region in Fig. 4	u_2'	u_1'	T_2'	T_1'	p'	No. of region in Fig. 4	u_2'	u_1'	T_2'	T_1'	p'
1	>0	>0	<0	>0	<0	9, 17	<0	>0	>0	<0	<0
2	>0	>0	<0	<0	<0	10, 22	<0	>0	>0	<0	<0
3, 16	>0	>0	<0	<0	<0	11, 20	<0	>0	>0	<0	<0
4, 23	>0	>0	<0	<0	<0	12, 27	<0	>0	>0	<0	<0
5, 14	>0	<0	<0	<0	<0	13	<0	<0	<0	<0	<0
6	<0	<0	<0	<0	<0	15, 24	<0	<0	<0	<0	<0
7, 19	<0	<0	<0	<0	<0	21, 28	<0	<0	<0	<0	<0
8, 18	<0	<0	<0	<0	<0	25, 26	<0	<0	<0	<0	<0

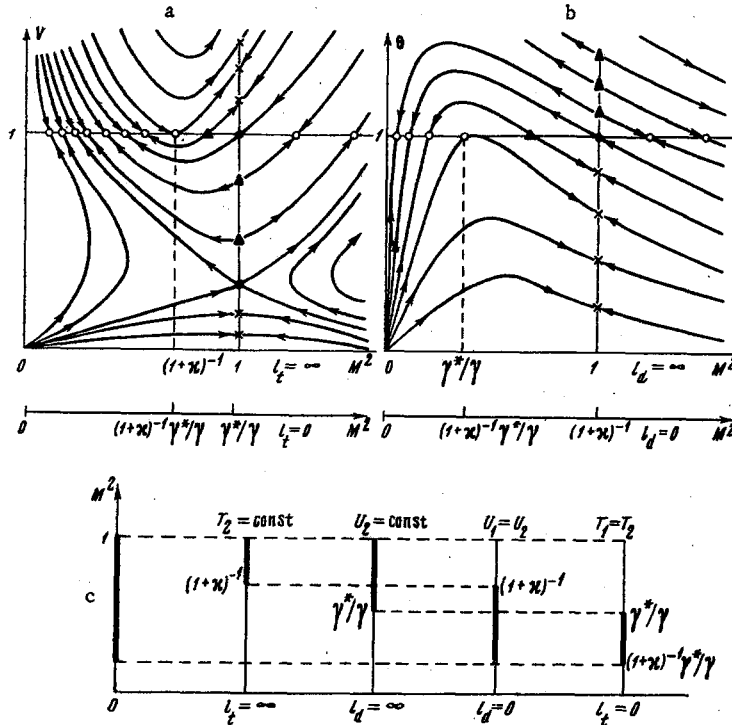


Fig. 6

For these conditions the system of equations (1.1)-(1.6) reduces to equations of the law of inversion of actions. They constitute a system of two equations for V and M_*^2 for $l_t=0$ or $l_t=\infty$:

$$(M_*^2 - 1) \frac{d \ln V}{dx} = \left[\kappa \frac{\gamma M_*^2}{V^2} (\Gamma - V) - (M_*^2 - 1) \right] \frac{V - 1}{l_d}$$

$$(M_*^2 - 1) \frac{d \ln M_*^2}{dx} = \kappa \frac{(\gamma - 1) M_*^2}{V^2} \left[\gamma \left(\frac{2}{\gamma - 1} + M_*^2 \right) (\Gamma - V) + (M_*^2 - 1) \right] \frac{(V - 1)}{l_d}$$

where $M_*^2 = \gamma/\gamma_* M^2$ for $l_t=0$ ($\theta=1$) and $M_*^2 = M^2$ for $l_t=\infty$ ($T_2 = \text{const}$) and, respectively, for θ and M_*^2 for $l_d=0$ or $l_d=\infty$, we have

$$(M_*^2 - 1) \frac{d \ln \theta}{dx} = \left[\kappa \frac{c}{c_p} \frac{(1 - \gamma M_*^2)}{\theta} - (M_*^2 - 1) \right] \frac{(\theta - 1)}{l_t}$$

$$(M_*^2 - 1) \frac{d \ln M_*^2}{dx} = \kappa \frac{c}{c_p} (1 + \gamma M_*^2) \frac{\theta - 1}{\theta l_t}$$

where $M_*^2 = (1 + \kappa) M^2$ for $l_d=0$ ($V=1$) and $M_*^2 = M^2$ for $l_d=\infty$ ($u_2 = \text{const}$).

The phase diagrams constructed for the equations just presented and the zones of unstable equilibria are shown in Fig. 6. Since the qualitative form of the plane diagrams for the cases $\theta=1$ and $T_2 = \text{const}$, on one hand, and $V=1$ and $u_2 = \text{const}$, on the other hand, coincides in pairs, we have depicted in Fig. 6 two diagrams with two equal scales along the abscissa axis.

The phase trajectories in Fig. 6a and b show the character of variation of the dimensionless velocity V and the temperature θ and the square of the Mach number M_*^2 along the length of the duct (in the direc-

tion indicated by the arrows on the curves). The equilibrium states for $\Theta = 1$ or $T_2 = \text{const}$ are represented by the straight line $V = 1$ (Fig. 6a) and the straight line $\Theta = 1$ (Fig. 6b) for $V = 1$ or $u_2 = \text{const}$; the line $M_*^2 = 1$ corresponds to the limiting states.

All trajectories in Fig. 6 begin from the limiting states at the entry into the duct — on the line $M^2 = 1$, or they start from the states of unstable equilibrium. The trajectories end at the points of stable equilibrium on the line $V = 1$ (Fig. 6a) or the line $\Theta = 1$ (Fig. 6b) or on the line $M_*^2 = 1$ at the points corresponding to the crisis at the exit from the duct.

It should be emphasized the limiting states are matched on these diagrams in the case $\Theta = 1$ by the equation $M^2 = \gamma^*/\gamma$ and in the case $V = 1$ by the equation $M^2 = (1 + \kappa)^{-1}$, while for $T^2 = \text{const}$ and $u_2 = \text{const}$, just as in the general case of a three-dimensional phase diagram, they are matched by the equation $M^2 = 1$.

In spite of the qualitative correspondence between the phase trajectories in a space and on a plane, there is no continuous limiting transition between them — from weakly nonequilibrium flows ($\Theta \approx 1$ or $V \approx 1$) to equilibrium flows ($\Theta = 1$ or $V = 1$) and also from the conditions $T_2 \approx \text{const}$ or $u_2 \approx \text{const}$ to the equalities $T_2 = \text{const}$ and $u_2 = \text{const}$. This manifests itself first of all in the difference of the dimensions of the zone of unstable equilibria (Fig. 6c). In the general case on the line $V = 1$ and $\Theta = 1$ this zone occupied the segment bounded by the inequalities $(1 + \kappa)^{-1}\gamma^*/\gamma < M^2 < 1$. On plane diagrams on the line $V = 1$, for $T_2 = \text{const}$, we have $(1 + \kappa)^{-1} < M^2 < 1$ and, for $l_t = 0$, respectively $(1 + \kappa)^{-1}\gamma^*/\gamma < M^2 < 1$. Analogously to this, on the line $\Theta = 1$, for $u_2 = \text{const}$, we have $\gamma^*/\gamma < M^2 < 1$ and, for $V = 1$, respectively $(1 + \kappa)^{-1}\gamma^*/\gamma < M^2 < (1 + \kappa)^{-1}$. This means that for $\Theta = 1$ the critical velocity is given by the quantity $u^* = a_T = \sqrt{\gamma^* p / \rho}$ (for $\kappa \gg 1$, $\gamma^* = (c_p + \kappa) / (c_v + \kappa) \rightarrow 1$ is equal to the isothermal velocity of sound), while for $V = 1$ the critical velocity is $u^* = a_u = \sqrt{\gamma p / (\rho_1 + \rho_2)}$ (since $(\rho_1 + \rho_2) / \rho_1 = (1 + \kappa)$). Introducing the conditional Mach number $M_*^* = u_1 / u^*$, we obtain in both cases the critical velocity $M_*^* = 1$.

The results obtained here allow us in principle to determine the behavior of a two-phase flow (i.e., the trends of variation of the velocity and temperature of the phases, the pressure, the Mach number, and others along the length of a duct of constant cross section) for any initial conditions. They contain all varieties of regimes of flow as well as indications of a qualitative imperfection of the spatial calculation simplifications (assumptions about the partial equilibrium $u_1 = u_2$ or $T_1 = T_2$), particularly for subsonic flows. The availability of a three-dimensional phase diagram can considerably facilitate the course of numerical calculations for particular conditions. Fixing these conditions, we can establish the possibility of practical realization and the boundaries of existence of the individual regimes. By means of the diagram we can carry out an analysis of discontinuous two-phase flows (having jumps) for the system considered and also predict certain conclusions about the motion of a two-phase flow in ducts of variable cross section, which is a case of practical importance, and finally, we can take into account phase transitions. These problems, however, necessitate a special investigation.

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